Sears Function and Lifting Surface Theory for Harmonic Gust Fields

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Sears' function for the lift on a two-dimensional airfoil induced by a harmonic gust field in an incompressible flow is reviewed. It is observed that a much simpler function can be defined by a transformation of the gust reference point from the airfoil midchord to its leading edge. The simplicity of the modified Sears function permits accurate interpolation for the large number of reduced frequencies required in a gust frequency response analysis. Approximations to the modified Sears function are discussed for use in preliminary analysis. The form of the transformation appropriate for interpolation of gust loads obtained from lifting surface theory for finite aspect ratio wings is then discussed. Both one- and two-dimensional gust fields are considered. It is shown by example calculations for compressible subsonic flow that the transformation required by lifting surface theory is analogous to that required by the two-dimensional incompressible case. It is concluded that the computational savings permitted by interpolation makes lifting surface methods economical for gust frequency response analysis. It is also concluded that the Doublet Lattice Method for subsonic flows has the versatility required for load calculations for both one- and two-dimensional gust fields.

Nomenclature

a = $(1 + k_L^2)^{1/2}$
b = semichord; b_r is reference value
c_l = section lift coefficient; c_l' is transformed value; $\Delta c_l'$ is the transformed lift coefficient for a narrow gust on a single strip
c_m = section moment coefficient; c_m' is transformed value
c_n = coefficient in n-term exponential approximation to Küssner function
$H(k)$ = modified Sears function
k = reduced frequency, $k = \omega b/V$; k_r is reference value based on b_r; k_L is value based on scale of turbulence
L = scale of turbulence
S = power spectral density; subscripts 1 and 2 denote values for one- and two-dimensional gust fields, respectively
t = time
V = freestream velocity
x = streamwise coordinate; x_0 is gust reference point in two-dimensional case; x_g is gust reference station in lifting surface analysis; x_{li} is coordinate of the leading edge of the ith strip centerline
γ_n = coefficient in exponent of n-term approximation to Küssner function
η,ξ,ρ = spanwise, chordwise, and radial separation distances, respectively, of cross-correlated points
θ = phase angle
λ = harmonic gust wavelength
$\Phi(k)$ = generalized Sears function
$\varphi(k)$ = Sears function
$\psi(\sigma)$ = Küssner function
σ = dimensionless time (or distance), $\sigma = Vt/b$; σ_g denotes standard deviation of gust vertical velocity
ω = angular frequency

Received September 6, 1968; revision received September 4, 1969. This paper summarizes work performed under sponsorship of the Independent Research and Development Program of the McDonnell Douglas Corporation. The authors wish to acknowledge T. P. Kalman and H. H. Croxen who implemented the methods on the computer.

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Introduction

ATMOSPHERIC gusts determine certain design requirements for aircraft structural strength and fatigue life. Design for both structural strength and fatigue life of an aircraft requires consideration of the random nature of atmospheric turbulence. The loading aspects are determined by power spectral density (PSD) methods whose basic ingredient is the frequency response characteristics of the aircraft in a harmonic gust field. The computation of response power spectra requires that the frequency response be found for a large number of gust wave lengths, say, 100. The aerodynamic loads for the frequency response analysis include those arising from the harmonic gust field and those resulting from the motion of the aircraft.

Early applications of PSD methods to subsonic aircraft assumed that strip theory was a valid approximation for three-dimensional lifting surfaces. Then, neglecting the effects of compressibility, the aerodynamic loads were obtained from Sears' solution[1] for the harmonic gust field and from Theodorsen's solution[2] for the response motion. Recent advances in subsonic oscillatory lifting surface theory[3,4] now permit a rigorous analysis of finite span and compressibility effects so that the traditional incompressible strip theory may be relegated to the status of a preliminary design method.

Naturally, the increased accuracy of the lifting surface methods is accompanied by a corresponding increase in computing costs. The requirement for analysis of so large a number of wave lengths (or reduced frequencies) would appear to make the use of lifting surface methods an exorbitant luxury were it not for the possibility of interpolation. The aerodynamic loads for motion are smoothly varying functions of reduced frequency and parabolic interpolation is adequate so that, e.g., 100 aerodynamic matrices for motion required for 100 frequency responses might be obtained by interpolating for 90 matrices among a set of 10 obtained in a direct calculation. Unfortunately, the gust loads are rapidly varying functions of reduced frequency and do not lend themselves to simple interpolation. However, a review of the two-dimensional Sears solution suggests a transformation that can be applied to three-dimensional lifting surface results to permit easy and accurate interpolation.

Review of Sears' Solution

The vector diagram of the Sears function, denoted by $\varphi(k)$ where k is the reduced frequency, is shown in Fig. 1. It is seen to be intricate in appearance especially at high frequency where it spirals about the origin. The amplitude of the spiral eventually decreases to zero, while the phase angle increases without limit as k approaches infinity. Three-dimensional lifting surface theory produces similar complicated vector diagrams for the lift at various stations across the wing span. In terms of wave length λ, the reduced frequency is given by $k = 2\pi b/\lambda$ where ω is the frequency, V is the velocity, and b is the semichord. The intricacy of $\varphi(k)$ will be seen to be a result of the coordinate system chosen by Sears, i.e., with the origin at midchord. The significance of the origin is that it is the reference point at which the gust velocity has zero phase angle. If we choose an arbitrary reference point at x_0 (positive downstream) then the lift on the airfoil is determined by the function

$$\Phi(k) = \varphi(k) \exp(ikx_0/b) \tag{1}$$

An appropriate choice for x_0 may be found by considering the asymptotic behavior of $\varphi(k)$ at high k. From the limiting values of the Bessel functions with which $\varphi(k)$ is expressed, we find

$$\lim_{k\to\infty} \varphi(k) = (2\pi k)^{-1/2} \exp[i(k - \pi/4)] \tag{2}$$

It is immediately evident that the phase angle of $\Phi(k)$ may be made to approach the definite limit of $-\pi/4$ if x_0 is chosen at the airfoil leading edge, i.e., at $x_0 = -b$. We, therefore, define a modified Sears gust function as

$$H(k) = \varphi(k) \exp(-ik) \tag{3}$$

This function is also shown in Fig. 1 and is seen to have some similarity to the Theodorsen function. The modified Sears function may also be obtained from a Fourier transformation of the Küssner function[5] because Küssner chose the leading edge of the airfoil as the reference point for gust penetration.

Approximations to the Modified Sears Function

As we have noted, frequency response analysis for preliminary design purposes will continue to utilize incompressible strip theory so approximations to $H(k)$ are still useful. Letting

$$H(k) = |H(k)| \exp[i\theta(k)] \tag{4}$$

we note that Liepmann[6] has suggested an approximation for the magnitude

$$|H(k)| \approx (1 + 2\pi k)^{-1/2} \tag{5}$$

and Fung[7] (p. 411) has suggested

$$|H(k)| \approx \{[a + k]/[a + (\pi a + 1)k + 2\pi k^2]\}^{1/2} \tag{6}$$

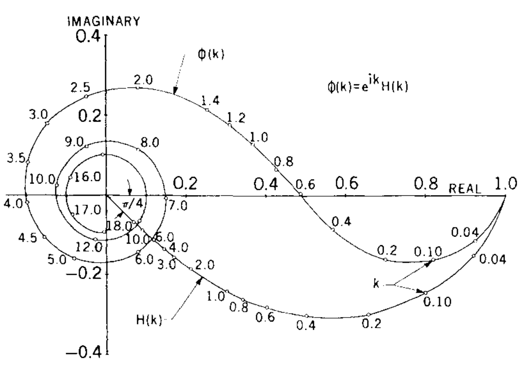

Fig. 1 The sears function $\phi(k)$ and the modified Sears function $H(k)$.

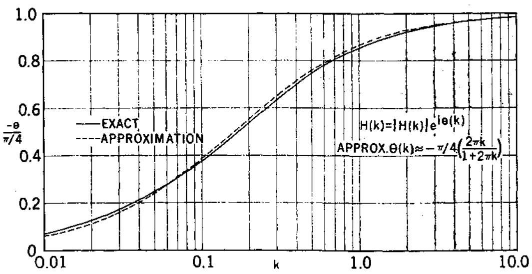

Fig. 2 The phase angle of the modified Sears function.

where $a = 0.1811$. A useful approximation for the phase angle has been found by the authors to be

$$\theta(k) \approx -(\pi/4)(2\pi k)/(1 + 2\pi k) \tag{7}$$

A comparison of Eq. (7) to the exact value is shown in Fig. 2. The approximation is seen to be accurate over a large range of reduced frequency.

The Fourier transform relationship between the modified Sears function and the Küssner function suggests another approximation. A number of exponential approximations to the Küssner function have been derived in the form

$$\psi(\sigma) \approx 1 - \Sigma c_n \exp(-\gamma_n \sigma) \tag{8}$$

where $\sigma = Vt/b$. The Fourier transformation leads to

$$H(k) \approx 1 - \Sigma c_n/(1 - i\gamma_n/k) \tag{9}$$

R. T. Jones[8] has obtained a three term exponential approximation with the constants $c_1 = 0.236$, $\gamma_1 = 0.058$, $c_2 = 0.513$, $\gamma_2 = 0.364$, $c_3 = 0.171$, and $\gamma_3 = 2.42$, while Sears and Sparks[9] have obtained a less accurate two term approximation with the constants $c_1 = c_2 = 0.500$, $\gamma_1 = 0.13$, and $\gamma_2 = 1.00$.

Applications to Lifting Surface Theory

In random gust response analysis, a common engineering practice is to assume a one-dimensional (streamwise) gust field. If the gust field is assumed to be homogeneous and isotropic the PSD is[10]

$$S_1(\omega) = [\sigma_g^2 L(1 + 3k_L^2)]/[2\pi(1 + k_L^2)^2] \tag{10}$$

where σ_g^2 is the variance of the gust vertical velocity, L is the scale of turbulence and $k_L = \omega L/V$ is the reduced frequency based on the scale of turbulence. The input PSD and the frequency responses of, say, a number of stresses caused by the harmonic gust field determine the PSD's of the stresses.

For the case of the one-dimensional gust, the loads calculated from lifting surface theory can be transformed for interpolation in a manner similar to that described for the Sears function. The lift and moment on each streamwise elemental strip at stations along the surface span (obtained from a lifting surface calculation such as the Doublet Lattice Method[4,11]) are transformed by shifting the gust field reference point from the original location x_g, say at the apex of the wing, to the leading edge of the streamwise strip. The transformed lift and moment coefficients for the ith strip are

$$c_{l_i}' = c_{l_i} \exp[ik_r(x_{l_i} - x_g)/b_r] \tag{11}$$

$$c_{m_i}' = c_{m_i} \exp[ik_r(x_{l_i} - x_g)/b_r] \tag{12}$$

where k_r is the reference reduced frequency of the (tapered) wing, b_r is the reference semichord, x_{l_i} is the streamwise coordinate of the leading edge of the ith strip centerline (corresponding to $x_0 = -b$ in the Sears function study), and x_g is the original gust field reference station. A similar transformation applies to the moment distribution. The moment was not considered in two-dimensional incompressible flow since the lift always acts at the one-quarter chord point in that case.

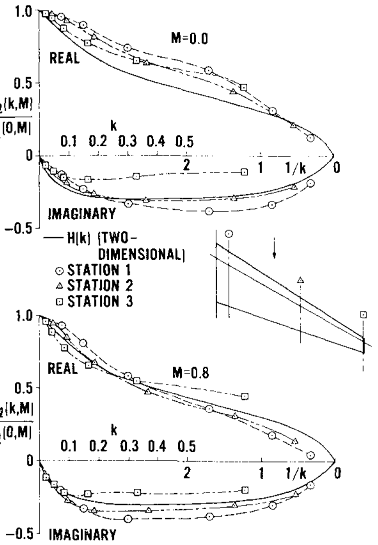

Fig. 3 Comparison of transformed harmonic gust lift coefficients on a typical jet transport wing with the modified Sears function.

After transforming the loads on each strip for all reduced frequencies, the interpolation may be carried out for all the additional required frequencies. Transforming the interpolated values back to the original gust field reference station by inverting Eqs. (11) and (12) provides the final form of gust aerodynamic loads for the frequency response analysis.

This transformation procedure was applied to a typical jet transport wing having an aspect ratio of 8.0, a quarter-chord sweep angle of 30°, and a taper ratio of 0.25. Two Mach numbers were considered: $M = 0.0$ and 0.8. The harmonic gust lift coefficients were obtained from the Doublet Lattice Method[11] for both Mach numbers and results for three spanwise stations (at the fractional location 0.0875, 0.565, and 0.990 of the wing semispan, respectively) are presented. The transformed lift coefficients are shown in Fig. 3§ as functions of reduced frequency based on the respective local chords and are normalized by their corresponding local steady-state values for comparison with the modified Sears function, $H(k)$. The transformed moment coefficients have not been shown but they exhibit essentially the same behavior. The comparison indicates that the transformation does indeed produce easily interpolated functions from lifting surface calculations and that, on a normalized basis, the transformed Sears function provides useful approximations of these results.

The foregoing assumption of a one-dimensional gust field in the streamwise direction is justified if the wing span is small relative to the atmospheric scale of turbulence, i.e., if the cross-PSD is essentially constant across the span. Lin[12] [Eq. (8A)] has given an expression for the cross-PSD of a homogeneous, isotropic, two-dimensional (axially symmetric) gust field. The expression can be generalized by replacing η, the spanwise separation distance, by ρ, the radial separation distance, and simplified to read

$$S_2(\omega,\rho) = \frac{S_1(\omega)}{1 + 3k_L^2}\left[3a^2\left(\frac{a\rho}{L}\right)K_1\left(\frac{a\rho}{L}\right) - \left(\frac{a\rho}{L}\right)^2 K_2\left(\frac{a\rho}{L}\right)\right] \tag{13}$$

where k_L is the reduced frequency based on the scale of turbulence L, $a^2 = 1 + k_L^2$, and the K_n are the modified Bessel functions of the second kind. Lin's expression contains a factor $\exp(-ik\xi/b)$, where ξ is the streamwise separation distance, that is not shown in Eq. (13) because it has been accounted for in the assumption of a harmonic gust over the wing chord. The ratio of the cross-PSD to auto-PSD, S_2/S_1, becomes a measure of the difference between the one-dimensional and the two-dimensional idealization of the gust field; the ratio is plotted in Fig. 4 as a function of ρ/L and k_L.

For the two-dimensional gust field, the frequency responses must be determined for narrow harmonic gusts traveling over each wing strip independently. The Doublet Lattice Method has the capability for this calculation. In the method, the wing is divided into streamwise strips of arbitrary width, and each strip is subdivided into a number of chordwise boxes of equal or smoothly varying chord lengths. The downwash at the control point of each box may be specified independent of all other boxes. It may also be specified on all boxes on a single strip independent of the other strips. A third alternative is to specify the downwash over the entire surface as was done for the one-dimensional gust. For the two-dimensional gust field, the harmonic streamwise variation in the downwash is specified on all boxes on a single strip. The incremental lifts and moments on all strips caused by gusts travelling over all strips may be summarized in a square matrix. (They may be summarized in a column matrix for the one-dimensional gust.) The transformation needed for interpolation is carried out as before for each element of the matrix. That is, the transformed incremental lift and moment coefficients for the ith strip for a gust on the jth strip are

$$\Delta c_{l_{ij}}' = \Delta c_{l_{ij}} \exp[ik_r(x_{l_i} - x_g)/b_r] \tag{14}$$

$$\Delta c_{m_{ij}}' = \Delta c_{m_{ij}} \exp[ik_r(x_{l_i} - x_g)/b_r] \tag{15}$$

This transformation procedure was also applied to the typical jet transport wing considered before. The detailed spanwise division of the wing into strips is shown in the upper half of Fig. 5. As an example of the use of the transformation given by Eq. (14), a gust is allowed to travel over the second wing strip $j = 2$. The frequency response of the local incremental lift coefficient for various spanwise stations is given in the lower half of Fig. 5. The locations of the five spanwise stations considered ($i = 2,3,4,7,14$) are the fractional locations 0.0875, 0.170, 0.265, 0.565, and 0.990 of the semispan, respectively. The normalization factor in this case is the same as that used in Fig. 3, i.e., the steady local lift-curve-slope. Using the same normalization parameter allows one to judge the effectiveness of allowing a gust to travel over just one strip as opposed to all strips (one-dimensional gust). Note that $\Delta c_{l_{22}}'$ (the incremental lift on strip 2 due to a gust impinging on strip 2) is 30% of the value obtained for the case of a gust impinging on all strips at once for $k = 0$.

Figure 5 shows that the transformation given in Eq. (14) reduces the frequency response to smooth, simple curves even for the off-diagonal terms ($\Delta c_{l_{ij}}'$ for $i \neq j$). Application of the transformation to other columns of the $\Delta c_{l_{ij}}'$ matrix (the case where a gust is allowed to impinge on all other strips) shows results similar to those presented in Fig. 5. The only substantial differences that occur are in the size of the off-diagonal terms in relation to the diagonal terms. For instance, when the gust travels over a strip of small sur-

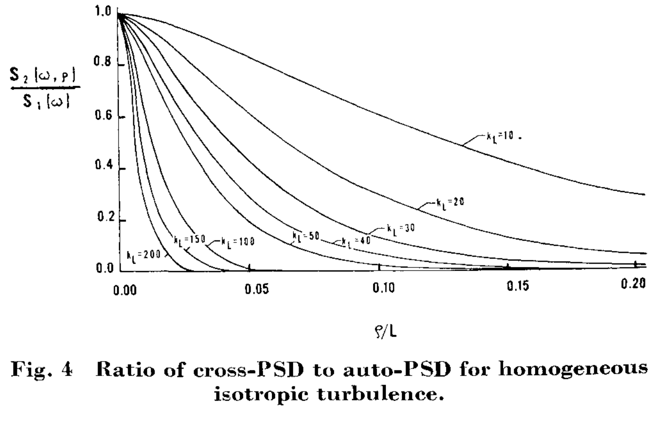

Fig. 4 Ratio of cross-PSD to auto-PSD for homogeneous isotropic turbulence.

§ Note the two scales for the reduced frequency to cover the complete range: $0 \leq k \leq 0.5$ and $2.0 \geq 1/k \geq 0$.

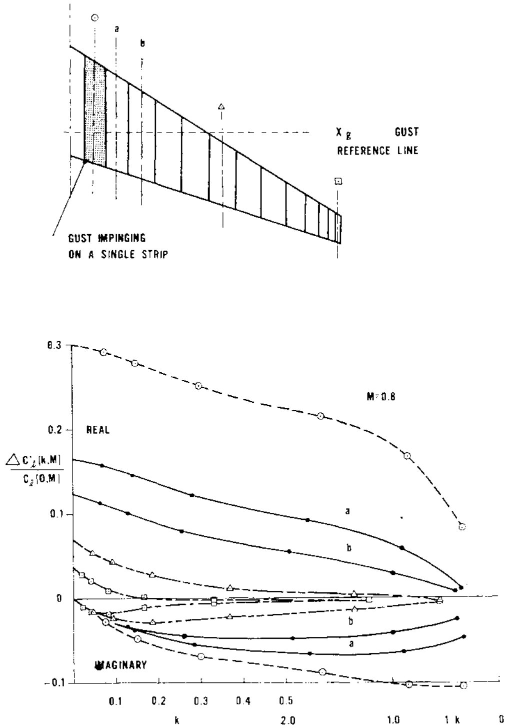

Fig. 5 Transformed lift coefficients at various wing stations due to a gust striking the second wing strip.

face area (e.g., near the wing tip) the effect of this gust diminishes very rapidly in the spanwise direction and renders the off-diagonal terms very small.

Concluding Remarks

The transformation suggested by the review of the Sears function has been shown to yield loads on finite aspect ratio wings that can reliably be interpolated for both one- and two-dimensional gust fields. The computational savings permitted by the interpolation shows that lifting surface methods can be utilized economically in gust frequency response analysis. The Doublet Lattice Method has also been demonstrated to have the versatility required for load determination in both one- and two-dimensional gust fields at subsonic speeds.

Although current engineering practice generally assumes a one-dimensional gust field, Fig. 4 suggests that the assumption may be overly conservative on very large aircraft. The degree of conservatism remains to be seen.

References

[1] Sears, W. R., "Some Aspects of Non-Stationary Airfoil Theory and Its Practical Applications," *Journal of the Aeronautical Sciences*, Vol. 8, No. 3, 1941, pp. 104–108.

[2] Theodorsen, T., "General Theory of Aerodynamic Instability and the Mechanism of Flutter," Rept. 496, 1935, NACA.

[3] Watkins, C. E., Woolston, D. S., and Cunningham, H. J., "A Systematic Kernel Function Procedure for Determining Aerodynamic Forces on Oscillating or Steady Finite Wings at Subsonic Speeds," TR R-48, 1959, NASA.

[4] Albano, E. and Rodden, W. P., "A Doublet Lattice Method for Calculating Lift Distributions on Oscillating Surfaces in Subsonic Flows," *AIAA Journal*, Vol. 7, No. 2, Feb. 1969, pp. 279–285; also "Errata," *AIAA Journal*, Vol. 7, No. 11, Nov. 1969, p. 2192.

[5] Küssner, H. G., "Das zweidimensionale Problem der beliebig bewegten Tragfläche unter Berucksichtigung von Partialbewegungen der Flussigkeit," *Luftfahrtforschung*, Vol. 17, 1940, pp. 355–361.

[6] Liepmann, H. W., "On the Application of Statistical Concepts to the Buffeting Problem," *Journal of the Aeronautical Sciences*, Vol. 19, No. 12, 1952, pp. 793–800, 822.

[7] Fung, Y. C., *An Introduction to the Theory of Aeroelasticity*, Wiley, New York, 1955.

[8] Jones, R. T., "The Unsteady Lift of a Wing of Finite Aspect Ratio," Rept. 681, 1940, NACA.

[9] Sears, W. R. and Sparks, B. O., "On the Reaction of an Elastic Wing to Vertical Gusts," *Journal of the Aeronautical Sciences*, Vol. 9, 1941, pp. 64–67.

[10] Liepmann, H. W., Laufer, J., and Liepmann, K., "On the Spectrum of Isotropic Turbulence," TN 2473, 1951, NACA.

[11] Stahl, B. et al., "Aerodynamic Influence Coefficients for Oscillating Planar Lifting Surfaces by the Doublet Lattice Method for Subsonic Flows Including Quasi-Steady Fuselage Interference," Rept. DAC-67201, Oct. 1968, McDonnell Douglas Corp.

[12] Lin, Y. K., "Transfer Matrix Representation of Flexible Airplanes in Gust Response Study," *Journal of Aircraft*, Vol. 2, No. 2, March-April 1965, pp. 116–121.